Average Unfairness in Routing Games

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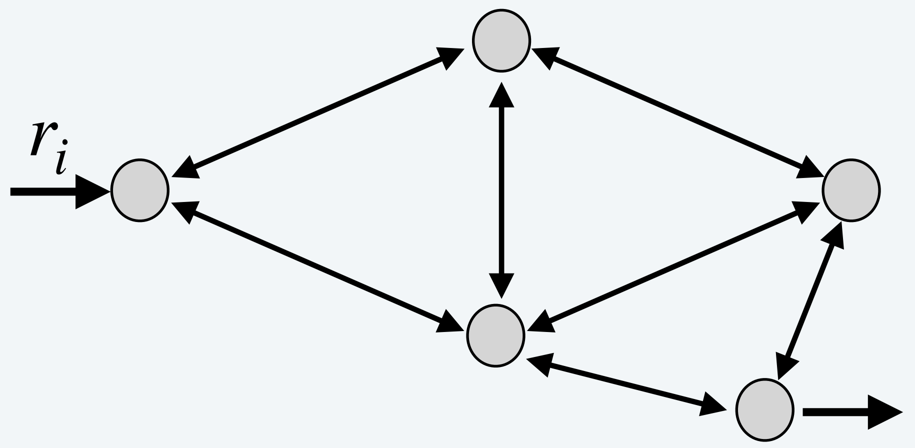
Fairness-Efficiency tradeoff

Routing games model selfish users choosing routes in a congested network.



Efficiency is measured with respect to aggregate system cost $C(f)$

Fairness accounts for the discrepancy in cost among users $U(f)$

Fairness-Efficiency tradeoff

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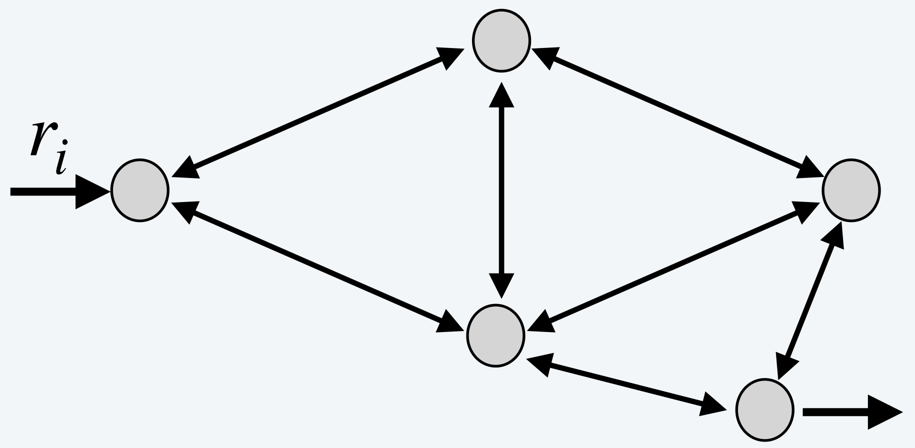

Efficiency is measured with respect to aggregate system cost $C(f)$

↓

The **System Optimal (SO)** flow minimizes the social cost but can be unfair.

⇄

Intrinsic tradeoff

Fairness accounts for the discrepancy in cost among users $U(f)$

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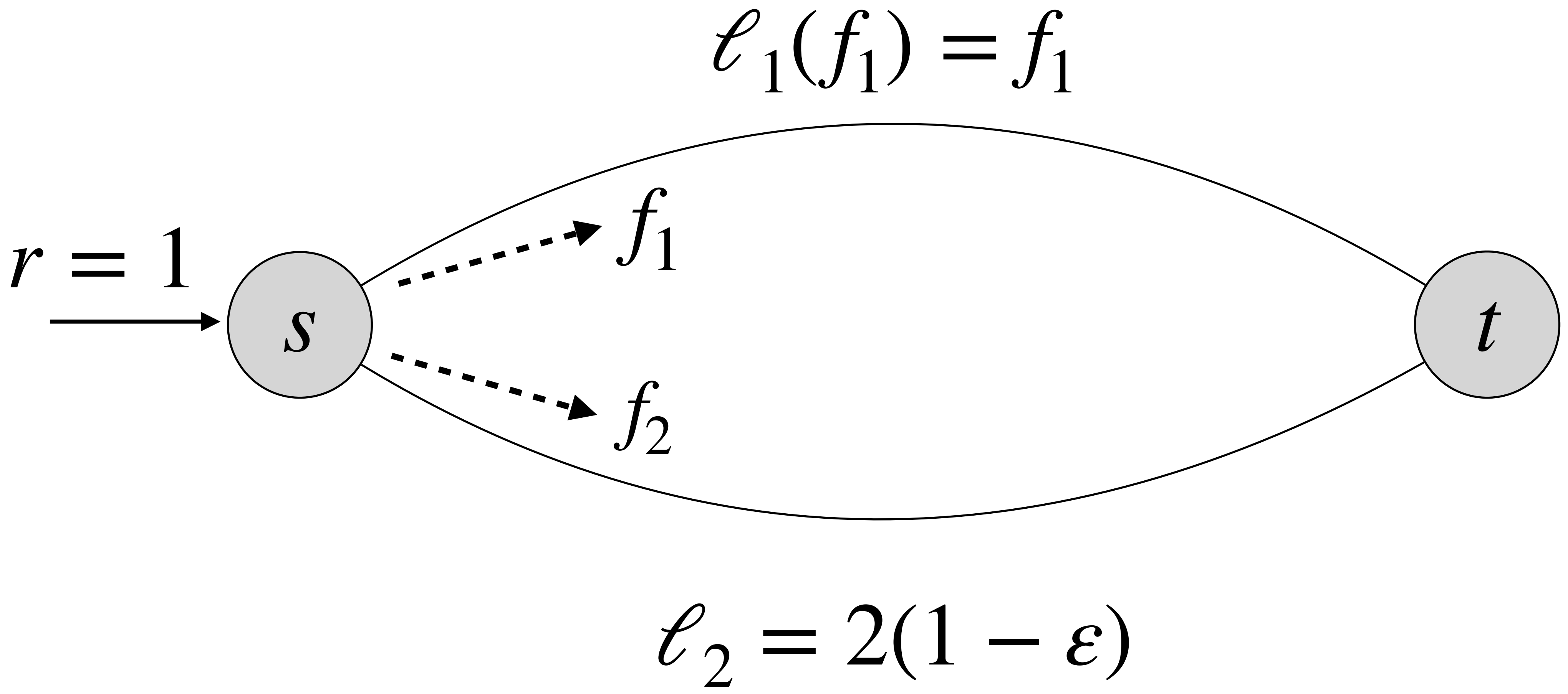
The **User Equilibrium (UE)** flow minimizes unfairness but can be inefficient.

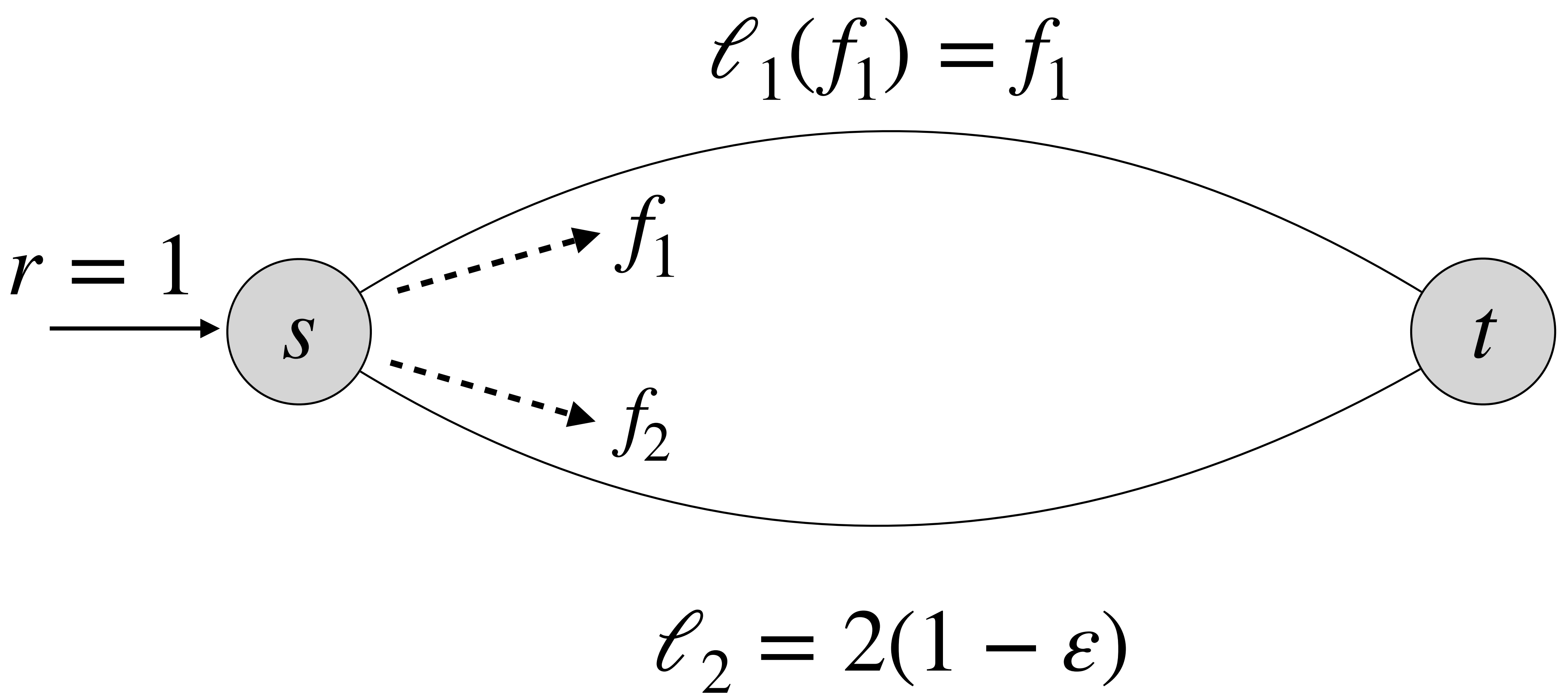
Fairness-Efficiency tradeoff



System cost $\quad C(f) = f_1\ell_1(f_1) + f_2\ell_2(f_2)$

Unfairness $\quad U(f) = \dfrac{\ell^{\max}}{\ell^{\min}}$

Fairness-Efficiency tradeoff



UE flow: $f_1 = 1 \quad f_2 = 0$

$C(f^{UE}) = 1$

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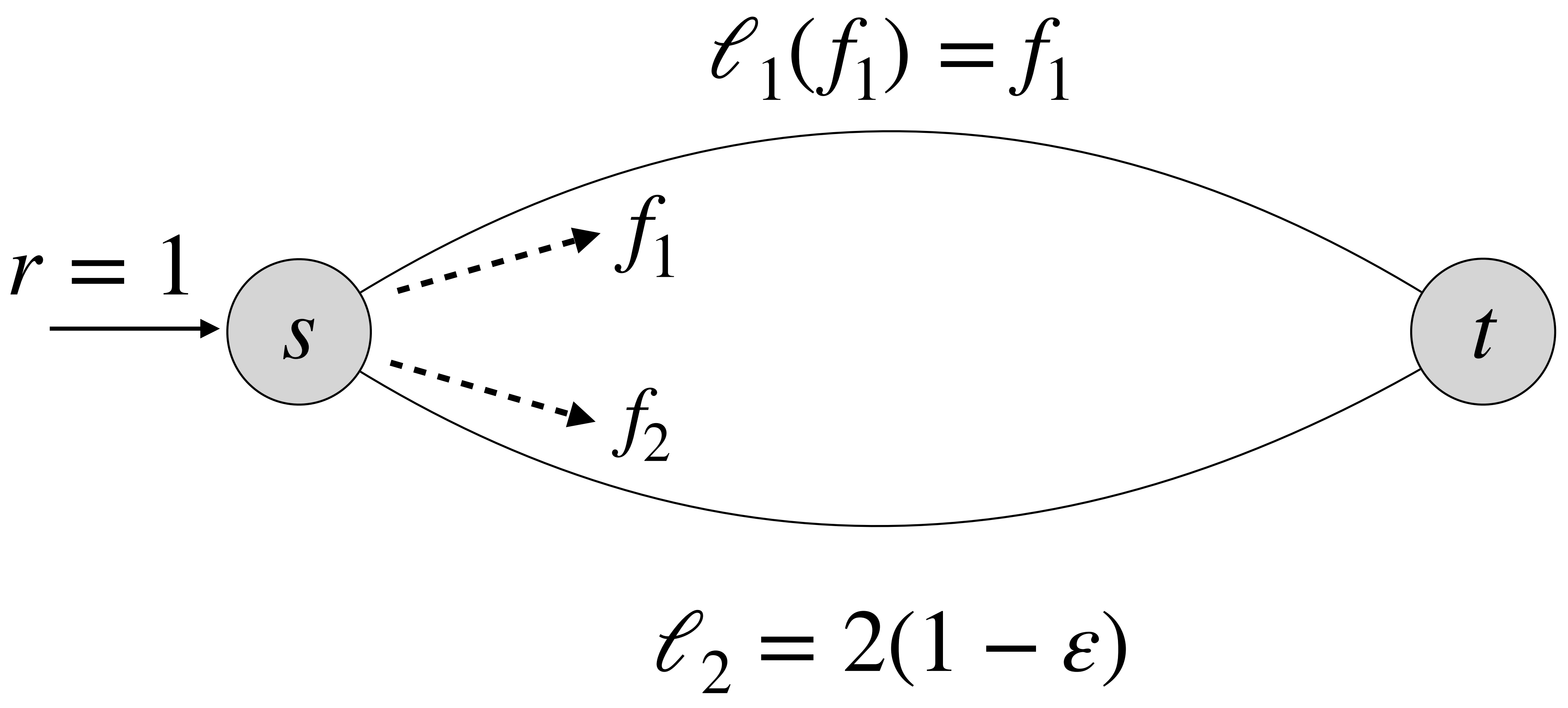

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UE flow: $f_1 = 1 \quad f_2 = 0$

$C(f^{UE}) = 1$ $\quad$ $U(f^{UE}) = 1$

SO flow: $f_1 = 1 - \varepsilon \quad f_2 = \varepsilon$

$C(f^\star) = 1 - \varepsilon^2$ $\quad$ $U(f^\star) = 2$

Questions of interest

What's the efficiency loss due to fairness?

Price of Anarchy/Stability $PoA/PoS = \dfrac{C(f^{UE})}{C(f^{\star})}$

[Roughgarden and Tardos, 2001], [Roughgarden, 2003], [Christodoulou, Koutsoupias & Spirakis, 2011]

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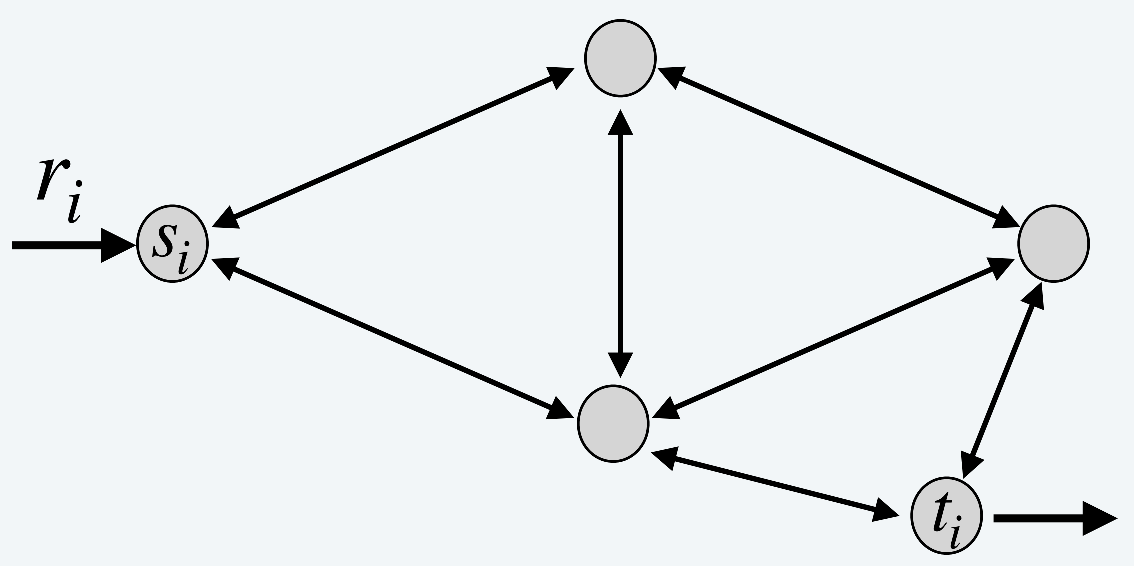
How do different unfairness measures compare and why does it matter?

[Jahn, et al., 2002], [Angelelli, et al., 2021], [Ferguson and Marden, 2021], [Jalota et al., 2023]

Routing game Model



Structure: Directed graph $G(V, E)$

System cost: Nonnegative, continuous, and nondecreasing latency $\ell_e(f_e)\ \forall e \in E$

Commodities: Set of trips $Z = \{(s_i, t_i, r_i)\}_{i=1}^K$

Action set for commodity i: all simple paths P_i from $s_i \rightarrow t_i$

Feasible flow: $f : P \rightarrow \mathbb{R}_+^{|P|} : \sum_{p \in P_i} f_p = r_i$

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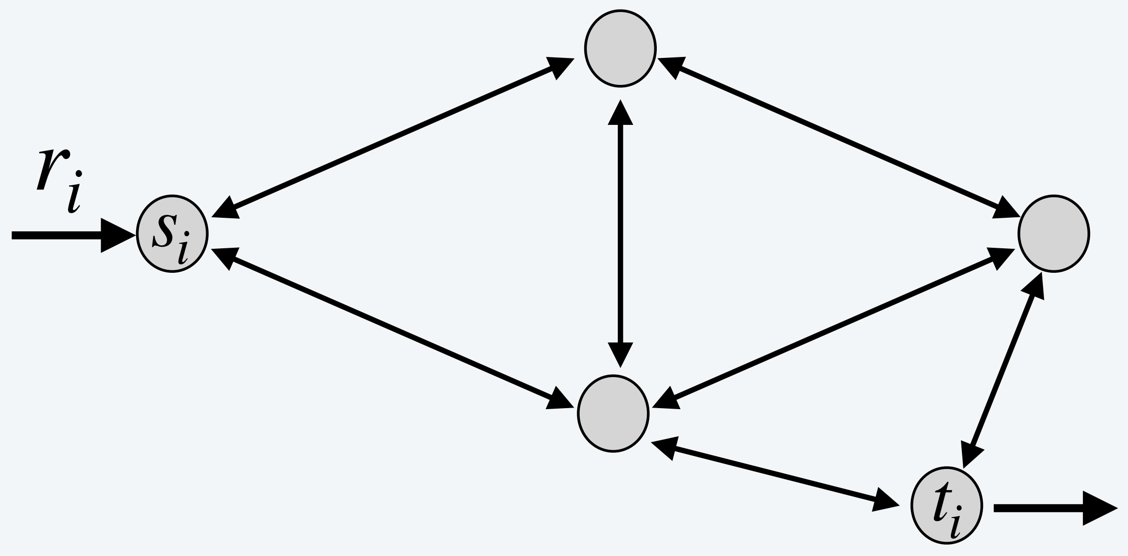

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User equilibrium flow f^{UE} All used paths have equal and minimal latency

For every commodity i and every pair of paths $p_1, p_2 \in P_i$ with $f_{P_1}^{UE} > 0$:

$$\ell_{P_1}(f^{UE}) \leq \ell_{P_2}(f^{UE})$$

System optimal flow $f^\star$ Minimizes the system cost

$$f^\star \in \arg\min_f C(f)$$

Unfairness measures

Let f be a feasible flow, $\ell_i^{\min}$ be the minimum latency of a path with positive flow and ℓ_i^{UE} be the minimal and common latency at UE, for commodity i.

Loaded unfairness [Jahn, et al., 2002; Correa, Schulz, & Stier-Moses, 2007]

$$U_i^L = \frac{\max_{p:f_p>0} \ell_p(f)}{\ell_i^{\min}}$$

Max-to-min latency ration among used paths. Captures **worst-case envy.**

UE unfairness [Jahn, et al., 2002; Roughgarden, 2002]

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Average unfairness (ours)

$$U_i^A(f) = \frac{\sum_{p \in P_i} f_p \ell_p(f)}{r_i \, \ell_i^{\min}}$$

Average-to-min latency ration among used paths. Captures **ex ante (average) envy.**

We aggregate unfairness across commodities by taking the the maximum

Tight bound on the PoE

Definition (Steepness) Let $\mathscr{L}$ be a class of differentiable latency functions for $\ell \in \mathscr{L}$ let $\hat{\ell} = \frac{d}{dx}\left(x\ell(x)\right)$

- For $\ell \in \mathscr{L}$, define steepness $\gamma(\ell) = \sup_{x>0} \frac{\hat{\ell}(x)}{\ell(x)}$
- The steepness of the class is $\gamma(\mathscr{L}) = \sup_{\ell \in \mathscr{L}} \gamma(\ell)$

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Theorem (Bound on the PoE) Let $\mathscr{L}$ be a class of differentiable latency functions containing all constants and denote $U(\mathscr{L}) = \sup_{\text{instances}} U(f^\star)$ then

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[Correa, Schulz, & Stier-Moses, 2007] [Ours]

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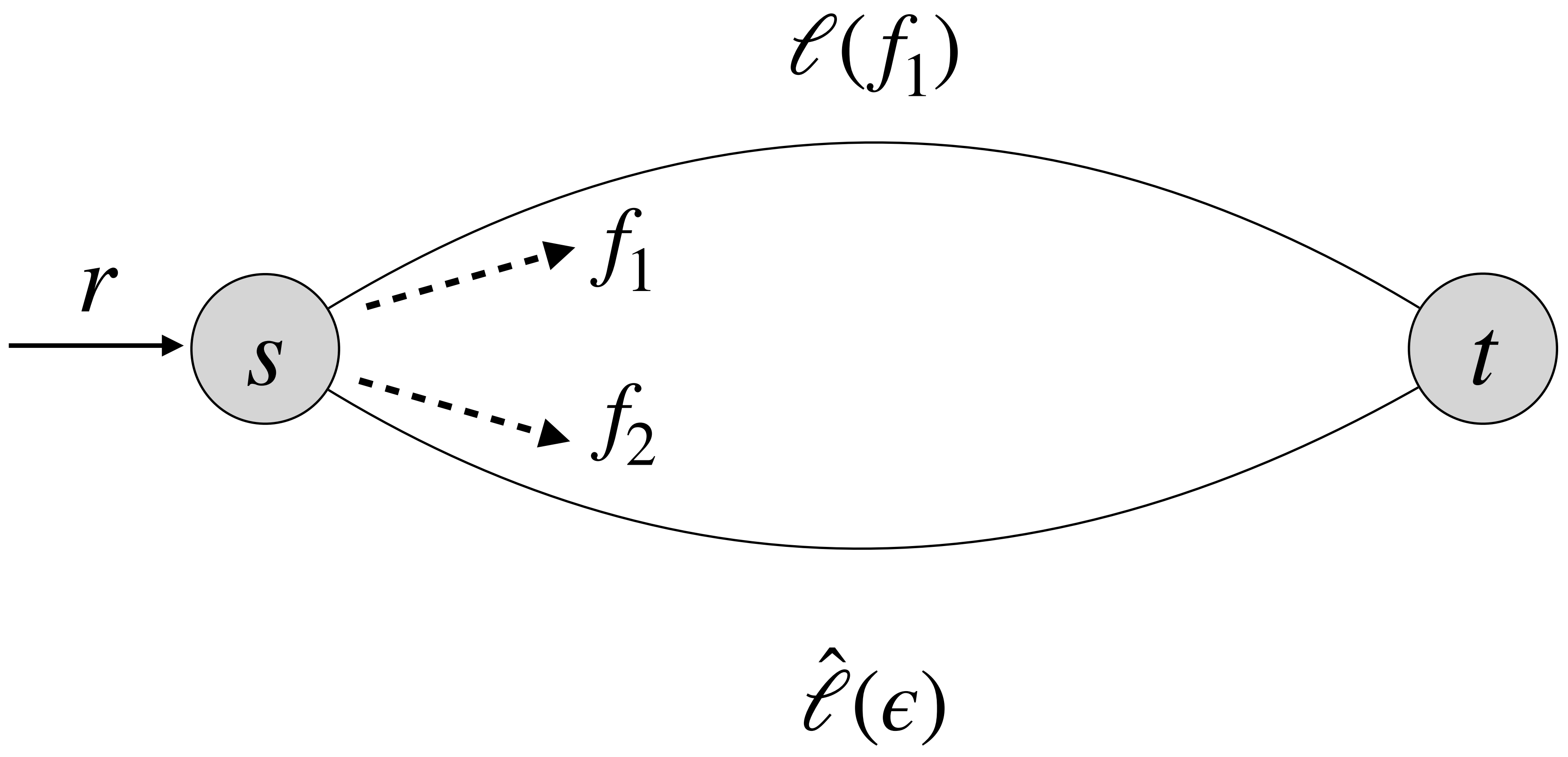
Proof (sketch)

Upper bound follows by $U^A(\mathscr{L}) \leq U^L(\mathscr{L}) \leq \gamma(\mathscr{L})$

Matching lower bound is attained in a Pigou network

For any $\delta \in (0, \gamma(\mathscr{L}))$ $\quad \exists \ell \in \mathscr{L}, \epsilon > 0:$

$$\hat{\ell}(\epsilon) \geq (\gamma(\mathscr{L}) - \delta)\ell(\epsilon)$$



Ordering of unfairness measures

Fix a game instance with a *single commodity* and let $f^\star$ be an optimal assignment.

$$\begin{tikzcd} U^L(f^\star) \arrow[r, "\geq"] \arrow[d, "\geq"'] & U^{UE}(f^\star) \arrow[dl, dashed, dash, "\text{No order}"] \\ U^A(f^\star) & \end{tikzcd}$$

Equality holds only if the assignment is totally fair w.r.t both measures

$$U^L(f^\star) = U^A(f^\star) = 1 \qquad \text{or} \qquad U^L(f^\star) = U^{UE}(f^\star) = 1$$

Why do we care about this order?

Constrained system optimum problem (CSO)

Minimize social cost

$$\min_{f} \quad C(f) := \sum f_p \ell_p(\boldsymbol{f})$$

$$\text{s.t.} \quad U(f) \leq 1 + \beta$$

Under unfairness constraints

Immediate observation

$$C(f^A) \leq C(f^L)$$

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When is this improvement strict?

f^L utilizes at least one path with minimal latency

$\implies U^A$ is continues around f^L

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Is this condition satisfied in general? No

How about in parallel link networks yes? Yes

Summary

- We propose a new average unfairness measure for routing games
- We extend existing PoE bounds to this new measure
- We provide analytical comparison of common unfairness measures
- We analyze the CSO solution under the various unfairness constraints

Future work

- Axiomatic approach towards defining unfairness measures
- Computationally tractable and normatively reasonable unfairness measures

Thank you!

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Tight bound on the PoE (Proof)

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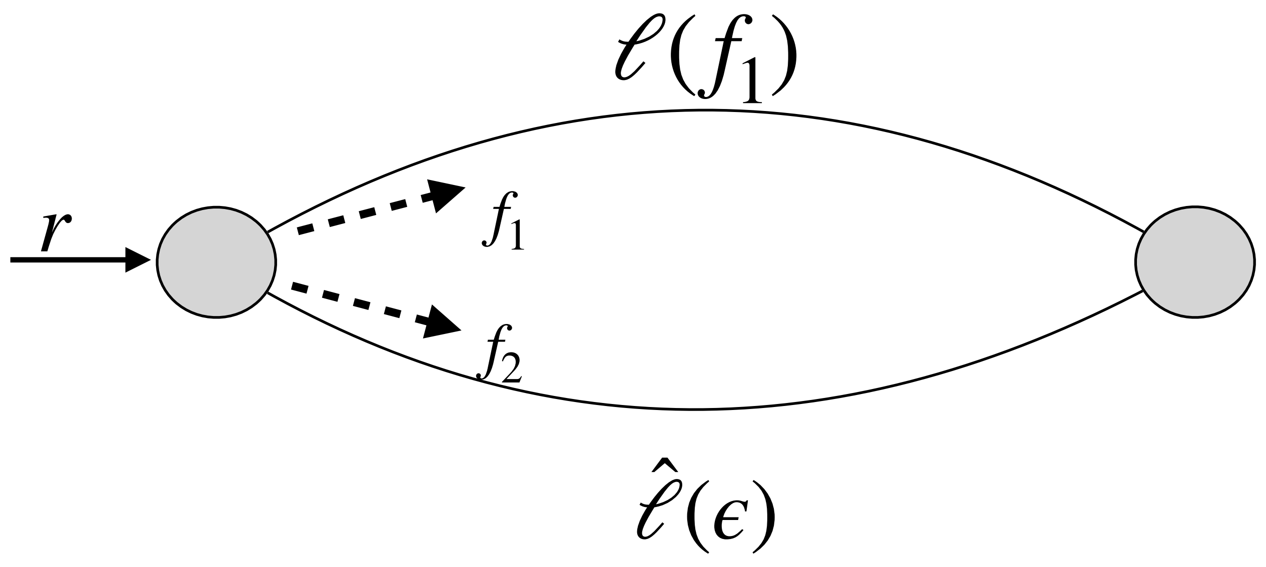

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Matching lower bound is attained in a Pigou network

- $f^\star$ is characterized by $\hat{\ell}(f_1) = \hat{\ell}(\epsilon) \implies f_1 = \epsilon$ is one solution
- The average unfairness satisfies $U^A(\mathscr{L}) \geq \lim_{r \to \infty} \dfrac{\epsilon \ell(\epsilon) + \hat{\ell}(\epsilon)(r - \epsilon)}{r\ell(\epsilon)} = \lim_{r \to \infty} \dfrac{\epsilon}{r} + \dfrac{\hat{\ell}(\epsilon)(r-\epsilon)}{r\ell(\epsilon)}$.
- The second term satisfies $\dfrac{\hat{\ell}(\epsilon)(r-\epsilon)}{r\ell(\epsilon)} \geq \dfrac{\hat{\ell}(\epsilon)(r-\epsilon)}{r\dfrac{\hat{\ell}(\epsilon)}{\gamma(\mathscr{L}) - \delta}} = (1 - \dfrac{\epsilon}{r})(\gamma(\mathscr{L}) - \delta) \xrightarrow[r \to \infty]{} \gamma(\mathscr{L}) - \delta$.

Why do we care about this order?

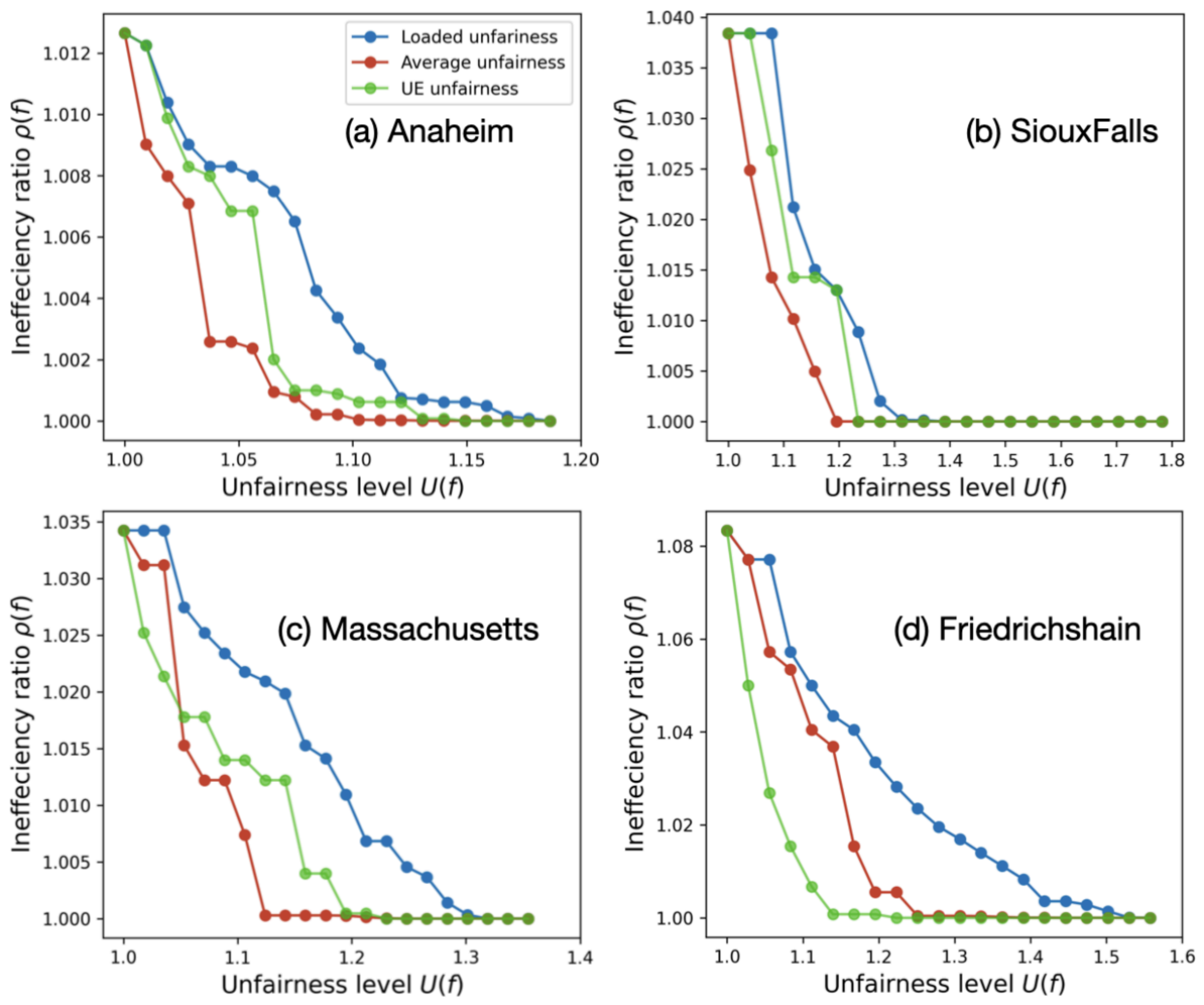

Figure 5: Comparison of CSO costs across varying unfairness levels under loaded (blue), average (red), and UE (green) unfairness measures for four benchmark networks.

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